

CONFINING BETHE–SALPETER EQUATION IN QCD

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ABSTRACT

We derive a confining $q\bar{q}$ Bethe–Salpeter equation starting from the same assumptions on the Wilson loop integral already adopted in the derivation of a semirelativistic heavy quark potential. We show that, by standard approximations, an effective meson squared mass operator can be obtained from our BS kernel and that, from this, by $\frac{1}{m^2}$ expansion, the corresponding Wilson loop potential is recovered, spin–dependent and velocity–dependent terms included. We also show, that, on the contrary, neglecting spin–dependent terms, relativistic flux tube model is reproduced.

In the paper presented by G.M. Prosperi ¹ it was shown how the properties of the Wilson loop integral (we assume Wilson area law and the straight line approximation; see eqs. (2)–(4)) can be used to obtain a confining Bethe–Salpeter equation from first principles. This result was accomplished neglecting the spin of the quarks. In this paper we show that it can be extended to the case of regular QCD with quarks with spin by defining an appropriate operator for the spin dependent part and using a second order formalism.

Even in this case the basic object is the quark–antiquark Green function

$$\begin{aligned} G_4(x_1, x_2, y_1, y_2) &= \frac{1}{3} \langle 0 | T \psi_2^c(x_2) U(x_2, x_1) \psi_1(x_1) \bar{\psi}_1(y_1) U(y_1, y_2) \bar{\psi}_2^c(y_2) | 0 \rangle = \\ &= \frac{1}{3} \text{Tr} \langle U(x_2, x_1) S_1(x_1, y_1; A) U(y_1, y_2) \tilde{S}_2(y_2, x_2; -\tilde{A}) \rangle \end{aligned} \quad (1)$$

where c denotes the charge-conjugate fields, U the path-ordered gauge string $U(b, a) = P_{ba} \exp \left(ig \int_a^b dx^\mu A_\mu(x) \right)$, S_1 and S_2 the quark propagators in the external gauge field A^μ , the tilde denotes transposition in the colour indeces.

Then, putting $(D_\nu = \partial_\nu - igA_\nu)$ $S(x, y; A) = (i\gamma^\nu D_\nu + m)\Delta^\sigma(x, y; A)$, we have $(\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu])$

$$(D_\mu D^\mu + m^2 - \frac{1}{2} g \sigma^{\mu\nu} F_{\mu\nu}) \Delta^\sigma(x, y; A) = -\delta^4(x - y), \quad (2)$$

and taking into account gauge invariance, we can write

$$G_4^{\text{gi}}(x_1, x_2; y_1, y_2) = (i\gamma_1^\mu \partial_{x_{1\mu}} + m_1)(i\gamma_2^\nu \partial_{x_{2\nu}} + m_2) H_4(x_1, x_2; y_1, y_2) \quad (3)$$

with

$$H_4(x_1, x_2; y_1, y_2) = -\frac{1}{3} \text{Tr} \langle U(x_2, x_1) \Delta_1^\sigma(x_1, y_1; A) U(y_1, y_2) \tilde{\Delta}_2^\sigma(x_2, y_2; -\tilde{A}) \rangle \quad (4)$$

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Now, we use the explicit resolution of (2) in terms of a path integral (Feynman-Schwinger representation; see (14) of Ref. ¹)

$$\begin{aligned}\Delta^\sigma(x, y; A) &= -\frac{i}{2} \int_0^\infty ds \exp \frac{is}{2} (-D_\mu D^\mu - m^2 + \frac{1}{2} g \sigma^{\mu\nu} F_{\mu\nu}) \\ &= -\frac{i}{2} \int_0^\infty ds \int_y^x \mathcal{D}z P_{xy} T_{xy} \exp i \int_0^s d\tau \left\{ -\frac{1}{2} (m^2 + \dot{z}^2) + g A_\rho(z) \dot{z}^\rho + \frac{g}{4} \sigma^{\mu\nu} F_{\mu\nu}(z) \right\}\end{aligned}\quad (5)$$

where the path integral is understood to be extended over all paths $z^\mu = z^\mu(\tau)$ connecting y with x and expressed in terms of a parameter τ with $0 \leq \tau \leq s$, \dot{z} stands for $\frac{dz(\tau)}{d\tau}$, the “functional measure” is assumed to be defined as $\mathcal{D}z = (\frac{1}{2\pi i \varepsilon})^{2N} d^4 z_1 \dots d^4 z_{N-1}$, P_{xy} and T_{xy} prescribe the ordering along the path from right to left respectively of the colour matrices and of the spin matrices.

On the other side, it is well known that, as a consequence of a variation in the path $z^\mu(\tau) \rightarrow z^\mu(\tau) + \delta z^\mu(\tau)$ respecting the extreme points, one has

$$\begin{aligned}\delta \quad & \{P_{xy} \exp ig \int_0^s d\tau \dot{z}^\mu(\tau) A_\mu(z)\} = \\ & = ig \int_0^s \delta S^{\mu\nu}(z(\tau)) P_{xy} \{ -F_{\mu\nu}(z(\tau)) \exp ig \int_0^s d\tau' \dot{z}^\mu(\tau') A_\mu(z(\tau')) \}\end{aligned}\quad (6)$$

with $\delta S^{\mu\nu}(z) = \frac{1}{2} (dz^\mu \delta z^\nu - dz^\nu \delta z^\mu)$. Then

$$\begin{aligned}T_{xy} \quad & \exp(-\frac{1}{4} \int_0^s d\tau \sigma^{\mu\nu} \frac{\delta}{\delta S^{\mu\nu}(z)}) (P_{xy} \exp ig \int_0^s d\tau' \dot{z}^\mu(\tau') A_\mu(z(\tau'))) \\ & = T_{xy} P_{xy} \exp ig \int_0^s d\tau [\dot{z}^\mu(\tau) A_\mu(z(\tau)) + \frac{1}{4} \sigma^{\mu\nu} F_{\mu\nu}(z(\tau))]\end{aligned}\quad (7)$$

and Eq.(6) can be rewritten as

$$\Delta^\sigma(x, y; A) = -\frac{i}{2} \int_0^\infty ds \int_y^x \mathcal{D}z P_{xy} T_{xy} \mathcal{S}_0^s \exp i \int_0^s d\tau \left[-\frac{1}{2} (m^2 + \dot{z}^2) + ig \dot{z}^\mu A_\mu(\bar{z}) \right] \quad (8)$$

with

$$\mathcal{S}_0^s = \exp \left[-\frac{1}{4} \int_0^s d\tau \sigma^{\mu\nu} \frac{\delta}{\delta S^{\mu\nu}(\bar{z})} \right] \quad (9)$$

In (9) it is understood that $\bar{z}^\mu(\tau)$ has to be put equal to $z^\mu(\tau)$ after the action of \mathcal{S}_0^s . Alternatively, it is convenient to write $\bar{z} = z + \zeta$, assume that \mathcal{S}_0^s acts on $\zeta(\tau)$ with $\delta S^{\mu\nu}(z) = \frac{1}{2} (dz^\mu \delta \zeta^\nu - dz^\nu \delta \zeta^\mu)$, and set eventually $\zeta = 0$.

Replacing (8) in (4) we obtain

$$\begin{aligned}H_4(x_1, x_2; y_1, y_2) &= \left(\frac{1}{2}\right)^2 \int_0^\infty ds_1 \int_0^\infty ds_2 \int_{y_1}^{x_1} \mathcal{D}z_1 \int_{y_2}^{x_2} \mathcal{D}z_2 T_{x_1 y_1} T_{x_2 y_2} \mathcal{S}_0^{s_1} \mathcal{S}_0^{s_2} \\ &\quad \exp \left(\frac{-i}{2} \right) \left\{ \int_0^{s_1} d\tau_1 (m_1^2 + \dot{z}_1^2) + \int_0^{s_2} d\tau_2 (m_2^2 + \dot{z}_2^2) \right\} \\ &\quad \frac{1}{3} \langle \text{Tr} P_\Gamma \exp(ig) \{ \oint_\Gamma d\bar{z}^\mu A_\mu(\bar{z}) \} \rangle\end{aligned}\quad (10)$$

where now $\bar{z} = \bar{z}_j = z_j + \zeta_j$ on Γ_1 and Γ_2 and $\bar{z} = z$ on the end lines x_1x_2 and y_2y_1 ; the final limit $\zeta_j \rightarrow 0$ being again understood.

Eq. (10) corresponds to Eq. (15) of Ref. ¹. Then, by using assumption (2)–(4) of Ref. ¹ for the Wilson loop integral, the recurrence identity (19) ¹ and proceeding in a similar way (apart from some technical complications) we can show ⁴ that the "second order" Green function $H_4(x_1, x_2; y_1, y_2)$ satisfies a Bethe-Salpeter type nonhomogeneous equation. From this we obtain the momentum space homogeneous equation

$$\Phi_P(k') = -i \int \frac{d^4k}{(2\pi)^4} \hat{H}_2(\eta_1 P + k') \hat{H}_2(\eta_2 P - k') \hat{I}(k', k; P) \Phi_P(k) \quad (11)$$

which is more appropriate for the bound state problem. In this equation H_2 stands for a colour independent one particle dressed propagator, $\eta_j = \frac{m_j}{m_1+m_2}$, P denotes the total momentum $p_1 + p_2$, k the relative momentum $\eta_2 p_1 - \eta_1 p_2$ ($q_j = \eta_j P + \frac{k+k'}{2}$ and in the CM frame $\mathbf{q} = \frac{\mathbf{k}+\mathbf{k}'}{2}$), $\Phi_P(k)$ is the ordinary Bethe-Salpeter wave function and

$$\begin{aligned} \hat{I}_{\text{pert}} = & 16\pi \frac{4}{3} \alpha_s \{ D_{\rho\sigma}(Q) q_1^\rho q_2^\sigma - \frac{i}{4} \sigma_1^{\mu\nu} (\delta_\mu^\rho Q_\nu - \delta_\nu^\rho Q_\mu) q_2^\sigma D_{\rho\sigma}(Q) \\ & + \frac{i}{4} \sigma_2^{\mu\nu} (\delta_\mu^\sigma Q_\nu - \delta_\nu^\sigma Q_\mu) q_1^\rho D_{\rho\sigma}(Q) + \frac{1}{16} \sigma_1^{\mu_1\nu_1} \sigma_2^{\mu_2\nu_2} (\delta_{\mu_1}^\rho Q_{\nu_1} - \delta_{\nu_1}^\rho Q_{\mu_1}) \times \\ & (\delta_{\mu_2}^\sigma Q_{\nu_2} - \delta_{\nu_2}^\sigma Q_{\mu_2}) D_{\rho\sigma}(Q) \} + \dots \end{aligned} \quad (12)$$

$\hat{I}_{\text{conf}} = \int d^3\mathbf{r} e^{i\mathbf{Q}\cdot\mathbf{r}} J(\mathbf{r}, q_1, q_2)$ with

$$\begin{aligned} J(\mathbf{r}, q_1, q_2) = & \frac{2\sigma r}{q_{10} + q_{20}} \left[q_{20}^2 \sqrt{q_{10}^2 - \mathbf{q}_T^2} + q_{10}^2 \sqrt{q_{20}^2 - \mathbf{q}_T^2} + \right. \\ & \left. + \frac{q_{10}^2 q_{20}^2}{|\mathbf{q}_T|} \left(\arcsin \frac{|\mathbf{q}_T|}{q_{10}} + \arcsin \frac{|\mathbf{q}_T|}{q_{20}} \right) \right] + \\ & + 2\sigma \frac{\sigma_1^{k\nu} q_{20} q_{1\nu} r^k}{r \sqrt{q_{10}^2 - \mathbf{q}_T^2}} + 2\sigma \frac{\sigma_2^{k\nu} q_{10} q_{2\nu} r^k}{r \sqrt{q_{20}^2 - \mathbf{q}_T^2}} + \dots \end{aligned} \quad (13)$$

In (12)–(13) we have set $q_1 = \frac{p'_1 + p_1}{2}$, $q_2 = \frac{p'_2 + p_2}{2}$, $Q = p'_1 - p_1 = p_2 - p'_2$, $D_{\rho\sigma}(Q)$ denotes the gluon free propagator and the center of mass system ($\mathbf{q}_1 = -\mathbf{q}_2 = \mathbf{q}$, $q_T^h = (\delta^{hk} - \hat{r}^h \hat{r}^k) q^k$) is understood. Notice that Eq. (12) corresponds to the usual ladder approximation in this second order formalism (differing from (17) ¹ only for the spin dependent terms.

From (11) by replacing $\hat{H}_2(p)$ with the free propagator $\frac{-i}{p^2 - m^2}$ and performing an appropriate instantaneous approximation on \hat{I} [consisting in setting $Q_0 = 0$, $q_{j0} = \frac{w'_j + w_j}{2}$ or $p_{j0} = p'_{j0} = \frac{w'_j + w_j}{2}$ or $k_0 = k'_0 = \eta_2 \frac{w'_1 + w_1}{2} - \eta_1 \frac{w'_2 + w_2}{2}$ and $P_0 = \frac{1}{2}(w'_1 + w_1 + w'_2 + w_2)$, with $w_j = \sqrt{m_j^2 + \mathbf{k}^2}$, $w'_j = \sqrt{m_j^2 + \mathbf{k}'^2}$] one can obtain ⁴ an effective mass

operator for the mesons (in the CM frame $\mathbf{P} = 0, P = (m_B, 0)$) $M = M_0 + V$ with

$$\langle \mathbf{k}' | V | \mathbf{k} \rangle = \frac{1}{(2\pi)^3} \frac{1}{4\sqrt{w_1 w_2 w'_1 w'_2}} \hat{I}_{\text{inst}}(\mathbf{k}', \mathbf{k}) + \dots \quad (14)$$

where the dots stand for higher order terms in α_s and σa^2 and kinematical factors equal to 1 on the energy shell have been neglected. Now, if we neglect in V the spin dependent terms and the coulombian one, we reobtain the hamiltonian of the relativistic flux tube model⁵ with an appropriate ordering prescription³. On the other side by performing a $\frac{1}{m^2}$ expansion we find the $q\bar{q}$ potential that, written in the representation space, reads

$$\begin{aligned} V &= \frac{4}{3} \frac{\alpha_s}{r} + \sigma r + \frac{4}{3} \frac{\alpha_s}{m_1 m_2} \left\{ \frac{1}{2r} (\delta^{hk} + \frac{r^h r^k}{r^2}) q^h q^k \right\}_W \\ &\quad - \frac{4}{3} i \alpha_s \left(\frac{1}{2m_1} \frac{\alpha_1 \cdot \mathbf{r}}{r^3} - \frac{1}{2m_2} \frac{\alpha_2 \cdot \mathbf{r}}{r^3} \right) + \frac{4}{3} \frac{\alpha_s}{2m_1 m_2} (\sigma_1 + \sigma_2) \cdot (\mathbf{r} \times \mathbf{q}) \\ &\quad + \frac{1}{3} \frac{\alpha_s}{m_1 m_2} \left[\frac{3(\sigma_1 \cdot \mathbf{r})(\sigma_2 \cdot \mathbf{r})}{r^5} - \frac{\sigma_1 \cdot \sigma_2}{r^3} \right] + \frac{4}{3} \frac{\alpha_s}{m_1 m_2} \frac{2\pi}{3} (\sigma_1 \cdot \sigma_2) \delta^3(\mathbf{r}) \\ &\quad - \frac{\sigma}{6} \left(\frac{1}{m_1^2} + \frac{1}{m_2^2} - \frac{1}{m_1 m_2} \right) \{ \mathbf{q}_1^2 r \}_W \\ &\quad - \frac{\sigma}{2} \left(\frac{\sigma_1}{m_1^2} + \frac{\sigma_2}{m_2^2} \right) \cdot \left(\frac{\mathbf{r}}{r} \times \mathbf{q} \right) - \frac{\sigma i}{2} \left[\frac{1}{m_1} \frac{\alpha_1 \cdot \mathbf{r}}{r} - \frac{1}{m_2} \frac{\alpha_2 \cdot \mathbf{r}}{r} \right] \end{aligned} \quad (15)$$

where now \mathbf{q} stands for the momentum operator and the symbol $\{ \}_W$ stands for the Weyl ordering prescription for momentum and position variables. Now, by performing a Foldy–Wouthuysen transformation with generator $S = \frac{i}{2m_1} \alpha_1 \cdot \mathbf{q} - \frac{i}{2m_2} \alpha_2 \cdot \mathbf{q}$ we end up with the $\frac{1}{m^2}$ potential which coincides with the Wilson loop potential³.

1. Conclusions

The kernel was constructed as an expansion in α_s and σa^2 and at the lowest order is given by equations (9)-(10).

As the analysis in terms of potentials show, the inclusion of terms in α_s^2 is essential for an understanding of the fine and the hyperfine structure. For what concerns the importance of σ^2 contributions some preliminary estimates⁴ performed in the relativistic flux tube context seem to indicate that this first correction should be of little significance for the spectrum in almost all cases.

Finally let us come to the problem of the type of confinement, which has been largely discussed in the literature. By this terminology it is usually meant the tentative assumption of a BS (first order) confining kernel of the instantaneous form $\hat{I}_{\text{conf}} = -(2\pi)^3 \Gamma \frac{\sigma}{\pi^2} \frac{1}{\mathbf{Q}^4}$. As well known, the above form of I with $\Gamma = 1$ was motivated by the fact that it reproduces the static potential σr and the spin dependent potential as obtained in the Wilson loop context. This choice, however, gets both into

phenomenological and theoretical difficulties: 1) it gives a first order velocity dependent relativistic correction to the potential which differs from the Wilson loop one ³ and does not seem to agree with the heavy meson data ⁷; 2) it does not reproduce straight line Regge trajectories ^{8,5}. Complementary objections can be moved to the form with $\Gamma = \gamma_1^0 \gamma_2^0$.

On the contrary, even if we have not yet attempted calculations directly with the kernel established in this paper, very encouraging results have been obtained in the context of the relativistic flux tube model ⁵, of the dual QCD ⁹ and of the effective relativistic hamiltonian ⁶, formalisms that are all strictly related to our one. Therefore the complicated momentum dependence appearing in (10) seems essential to understand both the light and the heavy meson phenomenology.

2. References

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